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SPECIAL SOLUTIONS OF THE STARK EQUATION

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Abstract. The Stark equation $-y'' + xy + q(x)y = \lambda y,\ 0 < x < \infty$ for potential $q(x)$ with the condition $\int_0^\infty |q(x)|\,dx < \infty$ is considered. The existence of solutions with asymptotics $Ai(x-\lambda)[1+o(1)]$ and $Bi(x-\lambda)[1+o(1)]$ for $x\to\infty$ is proved, where $Ai(x)$ and $Bi(x)$ are Airy functions of the first and second kind, respectively.

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1 Introduction and main results

In the paper Murtazin & Amangildin (1979) is considered the Stark operator

$$L = -\frac{d^2}{dx^2} + x + q(x)$$

on the positive semiaxis with the Dirichlet boundary condition at zero. The asymptotics of the spectrum for twice differentiable finite potentials $q(x)$ was also studied there. A similar result was obtained in the work Makhmudova & Khanmamedov (2020) for potentials $q(x)$ with a finite fourth moment that satisfy the condition $q(x) = o(x),\ x\to\infty$. The Stark operator with a finite moment is also of particular interest. On the other hand, it is known that (see (Avron & Herbst, 1977; Lin et al., 1989, 2017; Latifova & Khanmamedov, 2020; Korotyaev, 2017, 2018)) special solutions play an important role in the study of the spectral properties of the Stark operator.

The main content of this paper is related to the study of special solutions of the Stark equation

$$-y'' + xy + q(x)y = \lambda y,\ 0 < x < \infty. \tag{1}$$

We assume that the real function $q(x)$ satisfies the condition

$$\int_0^\infty |q(x)|\,dx < \infty. \tag{2}$$

It means that, we deal with the Airy functions $\mathrm{A}i(x)$ and $Bi(x)$, which are linearly independent solutions of equation (1) for $q(x) = 0, \lambda = 0$. Both of these functions are entire functions of order $3/2$ and type $2/3$ (see (Abramowitz & Stegun, 1964)). The Wronskian of these functions satisfies the following equality

$$W\{\mathrm{A}i(x), Bi(x)\} = \mathrm{A}i(x)Bi'(x) - \mathrm{A}i'(x)Bi(x) = \pi^{-1}.$$

We also introduce the functions φ and ρ

$$f_0(x,\lambda) = 2\pi^{\frac{1}{2}} Ai(x-\lambda), \tag{3}$$

$$g_0(x,\lambda) = 2^{-1}\pi^{\frac{1}{2}} Bi(x-\lambda), \tag{4}$$

which are solutions to equation (1) at $q(x)=0$. We will be interested in solutions $f(x,\lambda)$ and $g(x,\lambda)$ of equation (1) with asymptotics

$$f(x,\lambda) = f_0(x,\lambda)(1+o(1)),\ x\to\infty, \tag{5}$$

$$g(x,\lambda) = g_0(x,\lambda)(1+o(1)),\ x\to\infty. \tag{6}$$

Here, the existence of such solutions is proved. The results obtained can be used to study the spectrum of the Stark operator on the semi axis. The main results of this paper are the following theorems.

Theorem 1. *Let condition* (2) *be satisfied. Then, for each real value of* λ*, equation* (1) *has a unique solution* $f(x,\lambda)$ *that satisfies condition* (5).

Theorem 2. *Let condition* (2) *be satisfied. Then, for each real value of* λ*, equation* (1) *has a unique solution* $g(x,\lambda)$ *that satisfies condition (6).*

2 Proof of the theorems

Proof of Theorem 1. We use some properties of the Airy functions $\mathrm{A}i(z)$ and $Bi(z)$ for $|z|\to\infty$ (see (Abramowitz & Stegun, 1964))

$$\begin{aligned} &\mathrm{A}i(z) \sim (4\pi)^{-\frac{1}{2}} z^{-\frac{1}{4}} e^{-\zeta}\left[1+O\left(\zeta^{-1}\right)\right], \\ &Ai'(z) \sim -(4\pi)^{-\frac{1}{2}} z^{\frac{1}{4}} e^{-\zeta}\left[1+O\left(\zeta^{-1}\right)\right],\ |\arg z|<\pi, \end{aligned} \tag{7}$$

$$Bi(z) \sim \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} e^{\zeta}\left[1+O\left(\zeta^{-1}\right)\right],\ \ |\arg z| < \frac{\pi}{3}, \tag{8}$$

where $\zeta = \frac{2}{3} z^{\frac{3}{2}}$.

Consider the integral equation

$$f(x,\lambda) = f_0(x,\lambda) + \int_x^\infty F(x,t,\lambda)\, q(t)\, f(t,\lambda)\, dt, \tag{9}$$

equivalent to differential equation (1) with boundary condition (5), where

$$F(x,t,\lambda) = g_0(t,\lambda) f_0(x,\lambda) - g_0(x,\lambda) f_0(t,\lambda). \tag{10}$$

Without loss of generality, we assume that $\lambda \le 0$. As is known (Abramowitz & Stegun, 1964), the functions $Ai(x)$, $Bi(x)$, and hence, $f_0(x,\lambda)$, $g_0(x,\lambda)$ are non-negative at $x\ge 0$. Moreover, the function $f_0(x,\lambda)$ decreases monotonically, while $g_0(x,\lambda)$ increases monotonically. By virtue of relations (7), (8), the product $f_0(x,\lambda)\, g_0(x,\lambda)$ is bounded for $x\ge 0$:

$$|f_0(x,\lambda)\, g_0(x,\lambda)| \le C\ \ .$$

Let us investigate integral equation (9) by the method of successive approximations. For this purpose, we put

$$f^{(0)}(x,\lambda) = f_0(x,\lambda), \quad f^{(n)}(x,\lambda) = \int_x^\infty F(x,t,\lambda)\ \ q(t)\, f^{(n-1)}(t,\lambda)\, dt.$$

Then for $n=1$ we have

$$\left|f^{(1)}(x,\lambda)\right| \le \int_x^\infty |F(x,t,\lambda)|\ |q(t)|\ \left|f^{(0)}(t,\lambda)\right|\ dt \le$$

$$\le 2C\,|f_0(x,\lambda)| \int_x^\infty |q(t)|\,dt = 2C\,|f_0(x,\lambda)|\,\sigma(x)\,,$$

where $\sigma(x)=\int_x^\infty |\,q(t)|\ dt$. Similarly, for $n=2$, we obtain

$$\left|f^{(2)}(x,\lambda)\right| \le \int_x^\infty |F(x,t,\lambda)|\ |q(t)| \left|f^{(1)}(t,\lambda)\right|\ dt \le$$

$$\le 2C \int_x^\infty |F(x,t,\lambda)|\,|q(t)|\,|f_0(t,\lambda)\,|\,\sigma(t)\,dt \le$$

$$\le (2C)^2\,|f_0(x,\lambda)\,| \int_x^\infty |q(t)|\,\sigma(t)\,dt = \frac{(2C\sigma(x))^2}{2}\,|f_0(x,\lambda)\,|\ .$$

Further, by induction it is established that the following estimates are valid

$$\left|f^{(n)}(x,\lambda)\right| \le \frac{(2C\sigma(x))^n}{n\,!}\,|f_0(x,\lambda)\,|\ .$$

Taking into account the last estimates, we obtain the absolute and uniform convergence of the series $f(x,\lambda)=\sum_{k=0}^\infty f^{(k)}(x,\lambda)$ in the domain $x\in[0,\infty)$. The sum of this series obviously satisfies Eq. (9) and the estimate

$$|f(x,\lambda)| \le |f_0(x,\lambda)| \exp\{2C\sigma(x)\}\,. \tag{11}$$

Further, from (9), (10) we find that

$$f(x,\lambda) = 1 + \int_x^\infty \left[g_0(t,\lambda) - \frac{g_0(x,\lambda)\,f_0(t,\lambda)}{f_0(x,\lambda)}\right] f(t,\lambda)\,q(t)\,dt. \tag{12}$$

Using the monotonicity of the function $f_0(t,\lambda)$ and estimate (11), we obtain

$$\left|\int_x^\infty \left[g_0(t,\lambda) - \frac{g_0(x,\lambda)\,f_0(t,\lambda)}{f_0(x,\lambda)}\right] f(t,\lambda)\,q(t)\,dt\right| \le \int_x^\infty \left|g_0(t,\lambda)\,f_0(t,\lambda)\,e^{2C\sigma(t)}\right| |q(t)|\,dt+$$

$$+\,|g_0(x,\lambda)| \int_x^\infty \left|f_0(t,\lambda)\,e^{2C\sigma(t)}\right| |q(t)|\,dt \le Ce^{2C\sigma(x)} \int_x^\infty |q(t)|\,dt+$$

$$+\,|g_0(x,\lambda)\,f_0(x,\lambda)|\,e^{2C\sigma(x)} \int_x^\infty |q(t)|\,dt \le 2Ce^{2C\sigma(x)} \int_x^\infty |q(t)|\,dt \to 0,\ x\to\infty.$$

From this and (12), we obtain (5).

Proof Theorem 2. In this case we cannot use the integral equation of the form (9). Since the operator L generated by Eq. (1) and the boundary condition $y(0)=0$, is bounded from below, there exists $\lambda_0 \le 0$ such that $f(0,\lambda_0) \ne 0$. We put

$$\psi(x) = f(x,\lambda_0)\,. \tag{13}$$

Further, let a be so large that $\psi(x)=f(x,\lambda_0)\ne 0$ for $x\ge a$. Let us denote by $\varphi(x)$ the solution of the equation

$$-y'' + xy + q(x)\,y = \lambda_0 y,\ 0 \le x < \infty$$

with initial conditions $\varphi(a) = 0$, $\varphi'(a) = \psi^{-1}(a)$. It is easy to check that for $x \geq a$ this solution can be represented in the form

$$\varphi(x) = \psi(x) \int_a^x \psi^{-2}(t)\,dt.$$

Using then relations (5), (7) and the Lopital rule, we obtain

$$\lim_{x\to\infty} \frac{2\int_a^x \psi^{-2}(t)\,dt}{x^{\frac{1}{4}} e^{2x^{\frac{3}{2}}} \psi(x)} = 1.$$

Whence it follows that

$$\varphi(x) = \psi(x) \int_a^x \psi^{-2}(t)\,dt \sim \frac{1}{2} x^{\frac{1}{4}} e^{2x^{\frac{3}{2}}} \psi^2(x) \sim \frac{1}{2} x^{-\frac{1}{4}} e^{2x^{\frac{3}{2}}},\ x \to \infty. \tag{14}$$

Taking into account (8), (14), we have

$$\varphi(x) = g(x, \lambda_0) \sim g_0(x, \lambda_0),\ x \to \infty.$$

This completes the proof of Theorem 2.

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